Optimal Averaging Level Control

An optimal averaging level control problem is defined and solved for a surge tank system. Two averaging level controllers, the ramp controller and the optimal predictive controller are developed utilizing the optimal control policy. These controllers are compared with previously reported averaging level controllers for step and sinusoidal disturbances in inlet flow rate. The OPC results in improved disturbance filtering characteristics for most of the disturbances considered and is very simple to implement.

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SCOPE

Often the goal of liquid level control in a vessel is to provide outlet flow smoothing rather than tight level control. The primary control objective in this flow smoothing problem, referred to as averaging level control, is to use the surge capacity of the vessel to minimize the effect of inlet flow disturbances on downstream processing units.

In this paper the averaging level control problem is formulated as an optimization problem. The optimization objective is to find the outlet flow trajectory that results in the smallest maximum rate of change in the outlet flow (MRCO) subject to maximum and minimum height limits in the vessel not being exceeded. An optimal averaging level control law is derived for a surge tank system. This control law results in the smallest possible MRCO for a given maximum peak height (MPH) specification and a known step upset in the inlet flow.

Two averaging level controllers based on this optimal solution are presented. The first controller, referred to as the ramp controller, (RC) results from the fact that the optimal solution can be written in a nonlinear feedback form relating the level in the tank to the outlet flow rate. The ramp controller is compared with seven averaging level control schemes discussed in the literature for a variety of step disturbances and maximum peak height specifications.

The second controller, referred to as the optimal predictive controller (OPC) is a feedforward/feedback controller based on a predictive extension of the optimal control law. At each sampling instant the flow imbalance is measured, the future optimal control policy is determined, and the first control move is implemented. For this application it is assumed that the inlet flow rate can be measured and that there is negligible dead time associated with the disturbance. It may be possible to apply the control strategy to reboiler level control in the bottom of a distillation tower as well. For this application the unmeasurable net inlet flow to the reboiler (the difference between the incoming liquid and outgoing vapor flows) could be inferred from the filtered time derivative of the level in the tank and the liquid product flow.

CONCLUSIONS AND SIGNIFICANCE

An optimal averaging level control problem is defined and solved for a surge tank system. The solution is significant since it indicates the best attainable flow filtering subject to vessel constraints for a given step disturbance in inlet flow.

Two controllers based on this optimal solution are described and compared with previous approaches to the problem. A feedback implementation of the optimal solution, referred to as the ramp controller, produces significantly better flow filtering without undesirable flow overshoot. The tuning of this controller is straightforward and much easier than for some of the averaging level schemes previously proposed. One simply specifies the vessel constraints and the maximum expected step disturbance in the inlet flow. The resulting control law provides optimal filtering for this disturbance and good filtering for smaller disturbances as well.

The predictive implementation of the optimal solution provides near-optimal filtering for any size disturbance while insuring that vessel level constraints are not violated. To achieve the best possible filtering the optimal predictive controller (OPC) uses a large portion of the tank volume within the level constraints, and to minimize the overshoot in the outlet flow rate the controller is tuned to bring the level back to the desired steady state slowly. Thus, the OPC handles infrequent disturbances particularly well. Compared to the ramp controller, the nonlinear wide range controller, and the dynamic matrix control (DMC) approach, the OPC provides better flow filtering for inlet flow step disturbances of varying magnitudes. For high-frequency sinusoidal disturbances in the inlet flow rate the OPC and DMC give similar results in terms of flow filtering capabilities, but for lower frequency disturbances the DMC algorithm has much better filtering characteristics. This is primarily due to the fact that the loose level projection vector target area required to achieve "averaging" rather than "tight" level control in the DMC algorithm allows significant deviations in the level without changing the flow rate, whereas the OPC requires small changes in outlet flow even for small level deviations. Because of the difficulty in tuning and implementing the DMC controller, however, the OPC may provide a simple alternative for many averaging level control problems.

INTRODUCTION

Control of liquid inventory in a chemical plant is an important, basic problem. There are two competing objectives in such inventory control. First, the rate of change of flow from one vessel to another should be reasonably smooth to avoid upsetting downstream equipment. Second, minimum or maximum vessel level constraints must not be exceeded. Another objective that is important for averaging level control in a cascade of tanks is to minimize outlet flow overshoot, since this peak can be amplified as it moves down the cascade. The traditional approach to this liquid level control problem has involved the use of averaging controllers with proportional and integral modes. These averaging level controllers are deliberately detuned to produce a slow outlet flow response. However, they must be tuned tight enough to insure that maximum and minimum vessel level constraints are not violated.

Luyben and coworkers (Luyben and Buckley, 1977; Cheung and Luyben, 1979a,b, 1980), Shunta and Fehervari (1976), and Cutler (1982) have published results on averaging level control. Cheung and Luyben (1980) discuss seven different control design approaches to averaging level control. For both controller design and to compare various design approaches they used the criteria of the maximum rate of change in the outlet flow (MRCO) and the maximum peak height (MPH) after a step in inlet flow. The seven schemes that they compared are: proportional (P), proportional-integral (PI), proportional-lag (PL), nonlinear wide-range (NL), proportional-integral/proportional (PIP), dual-range integral-proportional (DRIP), and limited output change (LOC). One of the goals of Cheung and Luyben's study was to establish tuning charts for these controllers which would allow a designer to determine appropriate controller settings based on the MRCO and MPH specifications. In some cases a particular control structure could not simultaneously meet both specifications. Tuning charts are presented for the P, PI, and PL schemes; for the NL scheme tuning is problem-specific. With the PIP and DRIP schemes, trial and error tuning is required to meet the MPH criterion. Lastly, the LOC approach considers only the MRCO specification.

To bring the vessel level back to steady state the PI and NL schemes use a standard integral mode. This mode produces an undesirable overshoot in the exit flow. The PIP, DRIP, and LOC schemes all involve some type of logic switching. The PIP and DRIP schemes produce excellent flow filtering for small step upsets but they also exhibit flow overshoot for large step upsets. The PL scheme eliminates offset through feedforward action, while the P scheme tolerates offset. Cheung and Luyben concluded that the NL controller is more versatile than the P, PI, and PL controllers but that the resulting overshoot is a major drawback. They concluded that the LOC feature is desirable in digital implementation. Thus, the best controllers are those requiring either trial and error tuning or, in the case of the LOC scheme, one specification, MPH, is not considered directly.

More recently, Cutler (1982) demonstrated how a predictive control algorithm developed at Shell Oil Co. known as dynamic matrix control can be applied to systems with unusual dynamics. Predictive control refers to the use of a process model in a control algorithm to predict the future trajectory of the output variable. Control moves are calculated to optimize a performance index which considers the predicted output trajectory. A number of predictive control schemes have been discussed in the literature. These schemes include dynamic matrix control (Cutler and Ramaker, 1980; Prett and Gillette, 1980; Cutler, 1982), model algorithmic control (Mehra et al., 1981), internal model control (Garcia and Morari, 1982, 1984a,b), and schemes published by Marchetti et al. (1983a,b) and Chang and Seborg (1981). The work of Garcia and Morari provides an excellent general framework for understanding and analyzing predictive control approaches.

Cutler (1982) developed a DMC algorithm that can be applied to imbalanced or integrating systems. These are systems in which the controlled outputs are indicative of a system's heat or material balance. A step change in an input to such a system results in an output response that does not come to a constant steady state value but rather, a steady state ramp change in the output. As an example of such a system Cutler chose an averaging level control problem. Although this problem was chosen as an example for simulation studies, actual industrial applications of this DMC algorithm to averaging level control problems have not been reported.

Cutler compared conventional averaging level controllers (a detuned PI controller and an error-squared controller) with the DMC approach for level control in the bottom of a fractionator. The level response to a step change in feed flow rate to the tower includes a dead time and first-order dynamics before coming to a steady state ramp. The feedback controllers were detuned as much as possible to provide the maximum flow filtering while maintaining the liquid level in the vessel. The DMC algorithm does, however, incorporate feedforwarding of the inlet flow disturbance while the other controllers do not. A unique feature of Cutler's DMC approach to averaging level control is that it involves control to a level region rather than a specific setpoint. To obtain "averaging" rather than "tight" control, Cutler defines a level target area. This level projection target area is a funnel in time that is symmetrical about the level setpoint. If the projected level is within the funnel, the contribution to the projected error is set to zero. Figure 1, taken from Cutler's paper, shows simulation results for the level and outlet flow responses to an oscillatory disturbance in the feed flow rate. These simulation results indicate that the modified DMC algorithm provides much better flow filtering for this disturbance than the standard averaging level control techniques.

In this paper, a fundamentally different approach to averaging level control is taken. An optimum averaging level control problem is defined and solved for a surge tank system. For a given step in the inlet flow to the tank and a specified maximum peak height (MPH), the optimal control law provides maximum filtering by minimizing the MRCO. The incentive for choosing these optimization criteria is to provide an outlet flow trajectory that has the least impact on downstream processing units.

FORMULATION AND SOLUTION OF THE OPTIMAL CONTROL PROBLEM

Consider the surge tank system shown in Figure 2. The tank is initially at steady state with

$$h = h_s \tag{1}$$

$$q_i, q_o = q_s \tag{2}$$

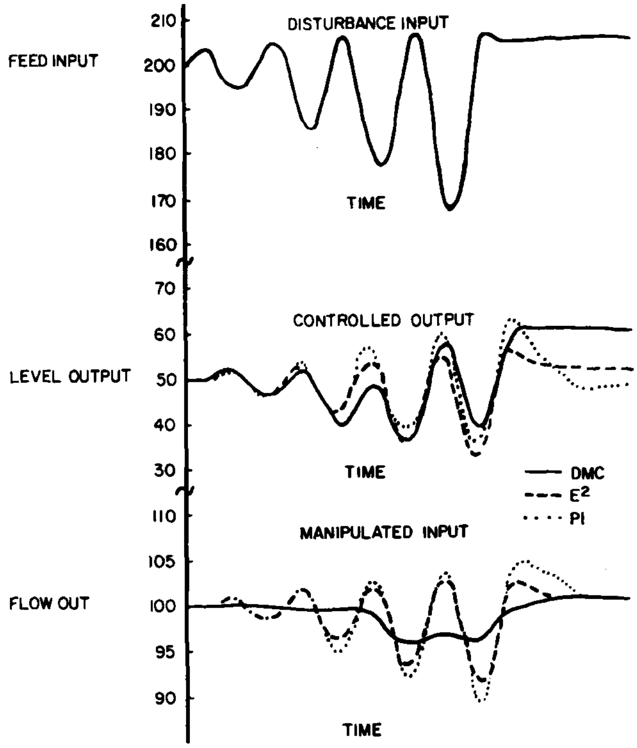

Figure 1. Fractionator simulation results (Cutler, 1982).

At $t = 0$ a step in feed to the tank of magnitude B occurs. The level in the tank is constrained by

$$h_{min} \leq h \leq h_{max} \tag{3}$$

The optimum control solution is developed assuming that B is positive and thus h_{max} equals the MPH. If h_{max} and h_{min} are symmetric around h_s, then the same control law can be used for both positive and negative errors. If h_{max} and h_{min} are not symmetric around h_s, then an identical development to that given below can be used to determine the optimal control law for negative errors. In this section h_{max} and h_{min} are assumed to be equidistant from h. Assuming that the liquid flowing into the tank is incompressible, then a mass balance can be used to relate h and q_o as

$$A\frac{dh(t)}{dt} = B + q_s - q_o(t) \tag{4}$$

where A is the cross-sectional area of the tank.

The optimization problem to be solved in the determination of an outlet flow trajectory that results in the smallest maximum rate of change in the outlet flow (MRCO) while satisfying Eqs. 1–4. As shown in the Appendix, the solution to this problem is a ramp function given by

$$q_o(t) = q_s + \alpha t \qquad t\epsilon[0, t_{max}] \tag{5}$$

with

$$\alpha = \frac{B^2}{2A(h_{max} - h_s)} \qquad t_{max} = \frac{2A(h_{max} - h_s)}{B}$$

FEEDBACK CONTROL

Formulation

It is possible to put the optimal solution in a feedback form relating deviations in q_o, $\hat{q}_o = q_o - q_s$, to deviations in h, $\hat{h} = h - h_s$. Substitution of Eq. 5 into Eq. 4 gives

$$\frac{A d\hat{h}}{dt} = B - \alpha t \tag{6}$$

Integration of Eq. 6 between 0 and t, with the initial condition of Eq. 1, results in:

$$\hat{h} = \frac{B}{A}t - \frac{\alpha t^2}{2A} \quad t\epsilon[0, t_{max}] \tag{7}$$

Eq. 5 can be used to eliminate t in Eq. 7 to give

$$\hat{h} = \frac{B}{A}\hat{q}_o - \frac{\hat{q}_o^{\,2}}{2A\alpha} \quad t\epsilon[0, t_{max}] \tag{8}$$

which is easily shown to be correct for $t > t_{max}$ as well, if q_o is fixed ($\hat{q}_o$ = constant = B). Finally, Eq. 8 can be solved to give the outlet flow as a function of h as

$$\hat{q}_o = B[1 - \sqrt{1 - \hat{h}/\hat{h}_{max}}] \tag{9}$$

where

$$\hat{h}_{max} = h_{max} - h_s \tag{10}$$

For negative values of B a corresponding development gives

$$\hat{q}_o = B[1 - \sqrt{1 - \hat{h}/\hat{h}_{min}}] \tag{11}$$

where

$$\hat{h}_{min} = h_{min} - h_s \tag{12}$$

Eqs. 9 and 11 correspond to the general control law

$$\hat{q}_o = B[1 - \sqrt{1 - |\hat{h}|/\hat{h}_m}] \tag{13}$$

where $\hat{h}_m$ is $h_{max} - h_s = h_s - h_{min}$.

Implementation

Several interesting facts about Eq. 13 can be noted. First, the optimal solution suggests a nonlinear proportional feedback control law. For a step disturbance of magnitude B this controller will result in the smallest MRCO. Since the size of the disturbance is unknown, one could simply specify a maximum expected disturbance and the MPH in Eq. 13, and use the resulting suboptimal control law. Even though the controller is suboptimal, it still compares favorably with the standard averaging level control algorithms.

It is usually desirable to ultimately bring the level back to the steady state setpoint after a disturbance in the inlet flow rate. Since the feedback controller is proportional in nature, it leads to level offset for step upsets. To eliminate offset, a standard feedback integral mode or a PL scheme (Luyben and Buckley, 1977) can be used. To avoid a large overshoot in the outlet flow rate and to minimize the effect of the integral action on the MRCO, the lag time constant in the PL scheme, τ_l, or the reset time in the integral

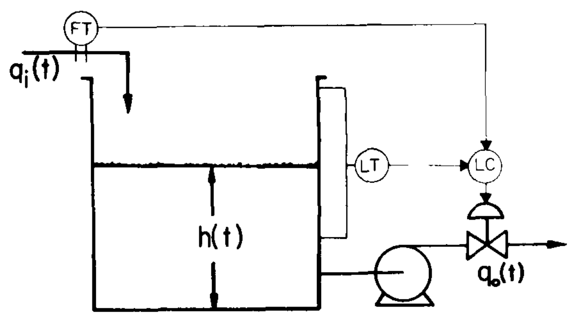

Figure 2. Surge tank system.

mode, τ_I, should be large. In the simulations carried out a value of $10t_{max}$ was found to produce acceptable results for both schemes. A larger value of τ_f or τ_I will result in a longer time period to return to the desired steady state level. If outlet flow overshoot can be tolerated then τ_f or τ_I can be reduced to bring the level back to the desired setpoint more quickly.

If the actual disturbance is larger than the maximum expected disturbance (i.e., poor choice for B), a different control law must be implemented since $\hat{h}$ may exceed $\hat{h}_m$, resulting in an imaginary term in Eq. 13. If $\hat{h}$ becomes larger than $\hat{h}_m$ the outlet flow rate must be set equal to the inlet flow rate to prevent the level from deviating any further. In our algorithm we set the outlet flow rate equal to the inlet flow rate when $\hat{h}$ reaches 99% $\hat{h}_m$ to strictly enforce the level limits. This action may result in a MRCO which is higher than that for the optimal solution, since the change in the outlet flow rate may be greater over this sampling period. The integral action will cause the outlet flow rate to exceed the inlet flow rate at the next sampling period, resulting in a decrease in the level. This implementation of the optimal solution with the integral mode will be referred to as the ramp controller (RC).

Comparison with Classical Schemes

Cheung and Luyben (1980) have presented results for various classical level control schemes for a surge tank system described in Table 1. This system is used here to determine how much improvement can be achieved using the ramp controller. Responses to a 100% step in the inlet flow were compared for four MPH specifications. These specifications are given in Table 2 together with the controller settings and resulting MRCO's for the P (proportional), PI (proportional-integral), PL (proportional-lag), NL (nonlinear wide-range), and RC (ramp controller) controllers. The controller gain for the P controller can be obtained directly from the MPH specification and the size of the step disturbance Cheung and Luyben 1979a). The PI and NL controllers were tuned for critical damping. At critical damping, the PI controller gain is also determined directly from the MPH specification and the size of the step disturbance. Tuning parameters for the NL controller were initially chosen from the tuning chart presented by Cheung and Luyben (1980) to minimize the MRCO. This chart was based on digital simulations of the NL controller with a sampling period of 0.1 min. Using a smaller sampling period, 0.001 min, we were able to reduce the MRCO even further. For example, the MRCO for case 1 was 3.43 m^3/min/min with a sampling period of 0.1 min, 2.83 using 0.01 min, and 2.78 with a sampling period of 0.001 min or smaller. Since the NL controller is compared with continuous P, PI, and PL controllers, a sampling period of 0.001 min was chosen for a consistent comparison. PL controller settings were obtained from tuning charts presented by Cheung and Luyben (1979b). The settings that appeared to minimize the MRCO were selected from the charts. Level and flow responses using the RC were simulated using a sampling period and integration time step of 0.001 min. The RC results were obtained using a standard integral mode to eliminate offset. The reset time was chosen as $10t_{max}$. Simulations of a PL scheme added to the optimal solution gave almost identical results to those shown for the standard integral mode. The percentage reductions in MRCO for the RC controller compared with the other types of controllers are given in Table 3. Compared with the P, PI, and PL controllers, the RC offers a substantial reduction in the MRCO. The advantage of the RC over the NL controller is smaller.

The RC was also compared to the PIP (proportional-integral/proportional) and DRIP (dual-range integral/proportional) schemes (Cheung and Luyben, 1980). With these schemes a split-range controller is used. For small errors one set of controller parameters is used while for large errors another type of control action is implemented. For a case presented by Cheung and Luyben (1980) (MPH = 0.8, 100% step disturbance) the MRCO of the optimal scheme is better than the DRIP scheme ($MRCO_{RC}$ = 0.86 m^3/min/min vs. $MRCO_{DRIP}$ = 1.0 m^3/min/min). A similar improvement over the PIP scheme can also be achieved. With the PIP and DRIP schemes some trial and error tuning is required, whereas tuning of the ramp controller is straightforward.

A direct comparison of the RC and the LOC (limited output change) schemes cannot be made because the LOC scheme does

TABLE 1. PARAMETERS FOR SURGE TANK SYSTEM (CHEUNG AND LUYBEN, 1980)*

Cross-sectional area of tank, A	1.0 m^2
Steady state level, h_s	1.0 m
Steady state flow, q_s	1.0 m^3/min
Range of level transducer, Δh	2.0 m
Range of valve, Δq	4.0 m^3/min
Tank holdup time, $\tau_V = \Delta hA/\Delta q$	0.5 min

*Original units converted to SI units; tank holdup time the same.

TABLE 2. COMPARISON OF AVERAGING LEVEL CONTROLLERS

Controller	Settings and MRCO*	Case 1	2	3	4
		Maximum Peak Height, MPH (m)			
		0.2	0.3	0.4	0.6
P	K_c	2.50	1.66	1.25	0.83
	MRCO	5.00	3.33	2.50	1.67
PI	K_c	1.84	1.23	0.92	2.17
(ξ = 1)	τ_I	1.09	1.63	2.17	3.26
	% OS**	13.50	13.50	13.50	13.50
	MRCO	3.68	2.45	1.84	1.22
PL	K_c	1.45	0.75	0.70	0.47
(K_F=1)	τ_F	1.00	1.00	2.00	3.00
	% OS	11.40	13.20	11.40	11.30
	MRCO	3.90	2.50	1.90	1.27
NL	K_{co}	1.32	0.75	0.65	0.37
(ξ =1)	K/τ	0.50	0.50	0.25	0.25
	I_o	1.52	2.67	3.08	5.41
	% OS	16.00	17.00	15.90	17.00
	MRCO	2.78	1.89	1.37	.93
RC	τ_T	4.0	6.0	8.0	12.0
	% OS	0.4	0.6	0.8	1.1
	MRCO	2.53	1.67	1.25	0.85
—	Optimal MRCO	2.50	1.67	1.25	0.83

* Maximum rate of change in outlet flow
** Percent overshoot in outlet flow

TABLE 3. PERCENT IMPROVEMENT* OF RAMP CONTROLLER OVER CLASSICAL SCHEMES

Case	Controller P	PI	PL	NL
1	97.6	45.5	54.2	9.9
2	97.0	45.0	47.9	11.8
3	96.9	45.0	49.6	7.9
4	96.5	43.5	49.4	9.4

* % Improvement = $\left(\frac{MRCO - MRCO_{RC}}{MRCO_{RC}}\right)100$

not use the MPH specification. With the LOC scheme one specifies the MRCO and accepts the resulting MPH. The transient responses presented by Cheung and Luyben (1980) for the LOC scheme when large reset times are used are very close to the optimal responses. By fixing $\dot{q}_o$ the LOC scheme gives rise to a ramp-type response in q_o which earlier was shown to be the best response to minimize the MRCO. The use of large reset times in conjunction with the LOC scheme is comparable to using a large τ_I (standard reset) or large τ_F (PL method) with the optimal scheme. The advantage of the RC over the LOC scheme is that the MPH specification can be taken into account directly in the controller design. More important, the approach presented in this paper shows exactly what the best attainable response is in terms of the MRCO. A predictive control extension of the RC approach is discussed below.

PREDICTIVE CONTROL

Formulation

The tank shown in Figure 2 is used for the derivation. At time t_o, the height is $h(t_o)$, the inlet flow is $q_i(t_o)$, and the outlet flow is $q_o(t_o)$. Assuming that the liquid is incompressible, a material balance on the tank gives

$$A\frac{dh}{dt} = q_i - q_o \tag{14}$$

Again, assume that the constraints given in Eq. 3 must be satisfied. In order to use predictive control one usually assumes that disturbances remain fixed in the future. Thus, q_i is assumed constant at $q_i(t_o)$ for $t > t_o$. At time t_o a flow imbalance, B, exists where B is given by

$$B = q_i(t_o) - q_o(t_o) \tag{15}$$

One can now ask how $q_o(t)$ should be manipulated for $t > t_o$ so that the maximum rate of change in the outlet flow, or MRCO, is minimized subject to Eqs. 3 and 14. This problem is essentially identical to the optimization problem solved previously (Eqs. 1–13). The only difference is that the system is not required to be at steady state at the time of the disturbance. Following the same arguments given above, and assuming B is positive, the optimal solution is a ramp in the outlet flowrate $q_o(t)$

$$q_o(t) = \frac{B^2(t - t_o)}{2A[h_{\max} - h(t_o)]} + q_o(t_o) \quad t > t_o \tag{16}$$

With predictive control one implements the optimal solution for a period of time, measures the disturbances and system state, and calculates a new optimum policy. Normally the predictive law is implemented discretely. In the case of Eq. 16 it is possible to derive a continuous law. If $q_o(t)$ is implemented over the time frame $t_o < t < t_o + \Delta t$, then rearranging Eq. 16 and substituting Eq. 15 gives

$$\frac{q_o(t) - q_o(t_o)}{\Delta t} = \frac{[q_i(t_o) - q_o(t_o)]^2}{2A[h_{\max} - h(t_o)]} \tag{17}$$

Taking the limit as $\Delta t \to 0$ gives the optimal predictive control law as

$$\dot{q}_o = \frac{(q_i - q_o)^2}{2A(h_{\max} - h)} \tag{18}$$

A derivation for the case where B is negative yields Eq. 18 as well with $h_{\min}$ in place of $h_{\max}$. Thus, the general optimal predictive law for no dead time in q_i can be written as

$$\dot{q}_o = \frac{(q_i - q_o)^2}{2A(h_m - h)} \quad \begin{cases} h_m = h_{\max} \text{ for } q_i > q_o \\ h_m = h_{\min} \text{ for } q_i < q_o \end{cases} \tag{19}$$

Note that the control law given in Eq. 19 is a feedback/feedforward controller and requires measurement of h, q_i, and q_o.

Implementation

There are two potential problems with the direct implementation of the control law given by Eq. 19. As with the feedback control law, level offset will be tolerated by the control system. The outlet flow will not change if the inlet and outlet flows are equal even though the level may not be at the desired steady state value. Secondly, if there is a difference in the steady state error between the inlet and outlet flow measurements, the outlet flow will change and thus cause the system to drift away from the steady state. This effect is shown in Figure 3 where at time $t = 0$ a bias of 0.08 m^3/min (2% full scale) has been added to the inlet flow measurement. If Eq. 19 is used, the difference in bias causes the outlet flow to increase. Since the inlet flow has not actually changed, the level begins to drop and the system moves away from its initial steady state.

To overcome these problems, the control law given by Eq. 19 has been modified as follows

$$q_o = \tilde{q}_o + K_c(h' - h_s) + \frac{K_c}{\tau_I}\int (h' - h_s)dt \tag{20}$$

where $\tilde{q}_o$ is determined from the equation

$$\dot{\tilde{q}}_o = \frac{(q'_i - \tilde{q}_o)^2}{2A(h_m - h')} \quad \begin{cases} h_m = h_{\max} \text{ for } q_i > q_o \\ h_m = h_{\min} \text{ for } q_i < q_o \end{cases} \tag{21}$$

The primed values, q'_i and h' refer to the process measurements which may contain noise and/or steady state error or bias. Since there may be a difference in the bias of the inlet and outlet measurements, h_m is determined by considering $\dot{h}'$ rather than by comparing q_i' and q_o'. If $\dot{h}' < 0$ then h_m is set equal to $h_{\min}$; if $\dot{h}' > 0$ then $h_m = h_{\max}$. To avoid toggling between $h_{\min}$ and $h_{\max}$, noise in the level measurements should be filtered before applying this criteria. If $\dot{h}'$is used to estimate the net inlet flow for reboiler level control it is also important to filter the noisy level signal.

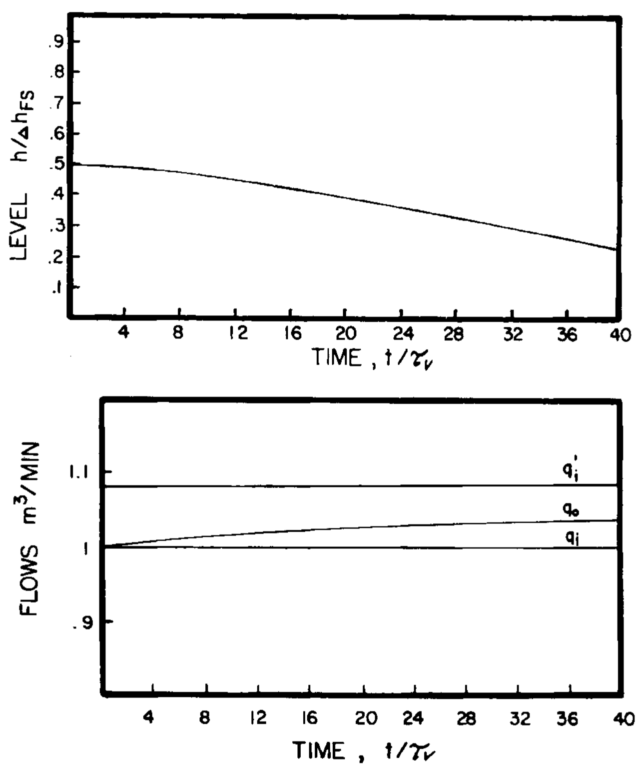

Figure 3. Effect of steady state error in inlet flow measurement on optimal control law (Eq. 19).

At steady state, Eqs. 14, 20, 21 require that $h = h_s$, $q_o = q_i$, and $q'_i = \tilde{q}_o$. These results give

$$\frac{K_c}{\tau_I}\int (h' - h_s)dt = -\left\{\begin{array}{l}\text{difference in bias}\\ \text{between inlet and}\\ \text{outlet flow measurements}\end{array}\right\} \quad (22)$$

Therefore, the integral term compensates for the difference in bias between the inlet and outlet flow measurements. Bias on the level measurement will result in a small steady state level offset, which will probably not be critical for the averaging level control problem.

Although the addition of proportional and integral modes in the modified control law, Eqs. 20 and 21, will slightly increase the maximum rate of change in the outlet flow rate, if K_c and τ_I are chosen properly this increase will not be significant. The effect of K_c and τ_I on the MRCO and the outlet flow overshoot for the test system described in Table 1 is shown in Table 4. These results are for a maximum peak height of 0.8 and a 50% step disturbance in the inlet flow with no noise or measurement bias. The time required to bring the level back to steady state after a step disturbance in the inlet flow will also depend on K_c and τ_I. When noise and bias are added to the flow measurements the response of the system becomes more sluggish. In addition, the MRCO may be slightly higher since the noisy inlet measurement is used in evaluating $\tilde{q}_o$ in Eq. 21. The implementation of the predictive control law given by Eqs. 20 and 21 is referred to as the optimal predictive controller or OPC.

Control performance will also depend on the sampling frequency or frequency of control action. A higher sampling frequency will result in a lower MRCO. In all of the simulations, the sampling period of the OPC was set equal to the sampling period of the controller with which it was being compared.

Comparison with Nonpredictive Strategies

The optimal predictive controller was compared with the best of the averaging level controllers discussed previously, i.e., the nonlinear wide-range controller (NL) and the ramp controller (RC). These two controllers provided better flow filtering for inlet flow step disturbances than the other controllers considered. The test system used for the comparison is described in Table 1. Sampling periods of 0.001 min were used for all three controllers.

The gain and reset time used for the OPC were determined by trial and error. For critical damping, the PI controller gain and reset time can be calculated from the maximum level and expected step disturbance (Cheung and Luyben, 1979a)

$$K_{c,\mathrm{PI}} = 0.736\,\frac{\Delta q_{inv}}{\hat{h}_{\max} A} \quad (23)$$

$$\tau_{I,\mathrm{PI}} = \frac{4\tau_v}{K_{c,\mathrm{PI}}} = 5.43\,\frac{\hat{h}_{\max} A}{\Delta q_i} \quad (24)$$

If the controller gain used for the OPC is 5–10% of $K_{c,\mathrm{PI}}$ the MRCO resulting from the optimal control law, Eq. 20 and 21, will not be increased significantly. The reset time can then be adjusted to provide the required settling time and appropriate outlet flow overshoot. In our simulations we found 1.4 $\tau_{I,\mathrm{PI}}$ was an appropriate reset time for the OPC. Thus reasonable PI tuning parameters for the OPC can be determined from the tuning parameters for a critically damped PI controller according to

$$K_{c,\mathrm{OPC}} = 0.05\,K_{c,\mathrm{PI}} \quad (25)$$

$$\tau_{I,\mathrm{OPC}} = 1.4\,\tau_{I,\mathrm{PI}} \quad (26)$$

Figures 4 and 5 show the responses of all three controllers to 100% and 10% steps in inlet flow rate for the case where the maximum peak height is 0.7. The gain and the reset time for the OPC, 0.046 and 3.0 min, respectively, were determined from Eqs. 25 and 26 for an expected disturbance of 100%. For the 100% step, all three controllers give very similar results (although the NL controller results in a larger outlet flow overshoot), but for the 10% step the OPC results in much better flow filtering since it uses more of the surge tank capacity. As a result, the MRCO is reduced significantly. If disturbances are infrequent and/or steps in the negative direction are just as likely as positive steps, then this reduction in MRCO is very beneficial. A problem could arise, however, if a larger step disturbance (in the same direction) hits the tank during the transient from a small upset, particularly if the timing was such that the tank level was already close to the maximum constraint. In this case, to avoid violating the constraint, the step disturbance must be passed more directly to the outlet flow rate, resulting in a substantial shock to downstream processing units. A solution to this problem would be to specify $h_{\max}$, or $h_{\min}$, as a function of the measured flow imbalance, $q_i - q_o$. In this way, a larger tank volume could be used for large disturbances while a smaller level deviation would result for smaller disturbances. Changing $h_{\max}$ and $h_{\min}$ as a function of the flow imbalance is similar to a gain scheduling approach to adaptive control. By changing $h_{\max}$ and $h_{\min}$ one should be able to achieve flow filtering characteristics in between the OPC and RC results shown in Figure 5 for small steps. Because the averaging level control problem becomes problem-specific when the size and sequence of disturbances is considered, methods to overcome this potential difficulty with the OPC are not analyzed in this paper.

COMPARISON WITH DYNAMIC MATRIX CONTROL

The OPC was compared to the DMC controller for the same system described in Table 1. The two controllers were compared for step disturbances as well as sinusoidal disturbances over a range of frequencies and amplitudes.

TABLE 4. EFFECT OF TUNING PARAMETERS (K_c, τ_I) ON PERFORMANCE

K_c	$\xi = 0.5$	$\xi = 0.25$	$\xi = 0.1$
(K_c = 0.0153 0.05 $K_{c,\mathrm{PI}}$	T_R = 32.7 MRCO* = 0.233 % OS** = 1.25	T_R = 8.17 MRCO = 0.233 % OS = 4.4	T_R = 1.31 MRCO = 0.233 % OS = 13.9
(K_c = 0.0307) 0.10 $K_{c,\mathrm{PI}}$	T_R = 16.3 MRCO = 0.248 % OS = 2.2	T_R = 4.07 MRCO = 0.248 % OS = 7.9	T_R = 0.652 MRCO = 0.259 % OS = 22.6
(K_c = 0.0967) 0.25 $K_{c,\mathrm{PI}}$	T_R = 6.52 MRCO = 0.367 % OS = 3.6	T_R = 1.63 MRCO = 0.443 % OS = 14.2	T_R = 0.261 MRCO = 0.400 % OS = 35.5

* Maximum rate of change in outlet flow
** Percent overshoot in outlet flow

Tuning the DMC Controller

Tuning the DMC controller presented by Cutler (1982) was much more involved than tuning the OPC since there were essentially six parameters that needed to be determined: the sampling period, the output horizon (number of future sampling periods considered), the input horizon (number of future control moves considered), the penalty on changes in the outlet flow rate, the two parameters associated with the projection vector target area or funnel. First, controller tuning parameters that would allow the level to reach a maximum peak height of 0.7 for 100% step in inlet flow were determined by trial and error. As can be seen in Figure 6, the performance of the DMC controller depends on the tuning parameters used in the simulation. A sampling period of 0.05 min, or 10% of the tank holdup time, was chosen to allow a number of control moves before the tank overflows for a 100% step in inlet flow. The tuning parameters associated with simulation *A* resulted in the lowest MRCO. For this simulation the total output horizon 0.5 min, is rather small. With these parameters, the DMC algorithm

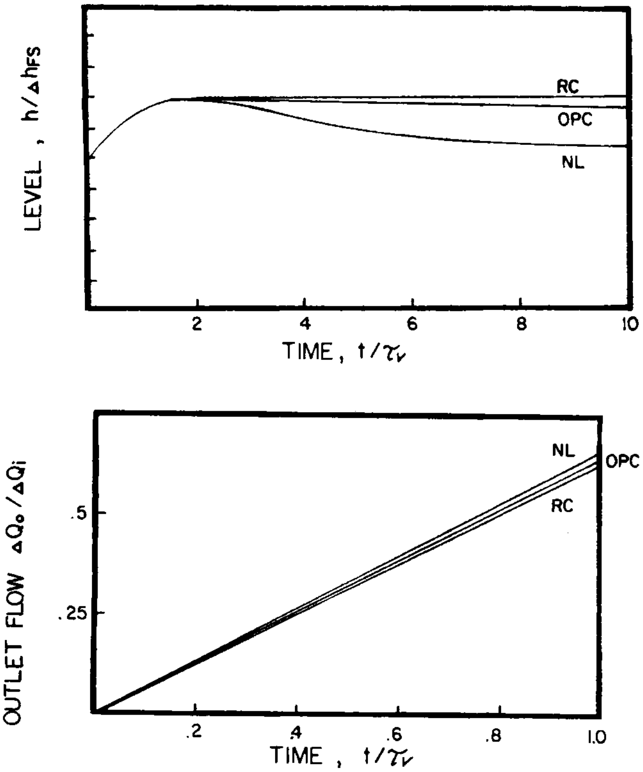

Figure 4. Comparison of averaging level controllers. MPH = 70%; 100% step in inlet flow.

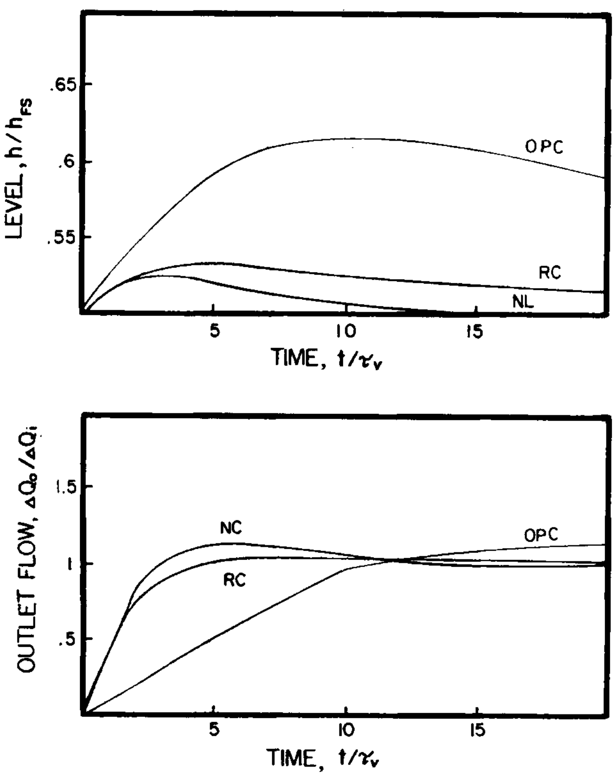

Figure 5. Comparison of averaging level controllers. MPH = 70%; 10% step in inlet flow.

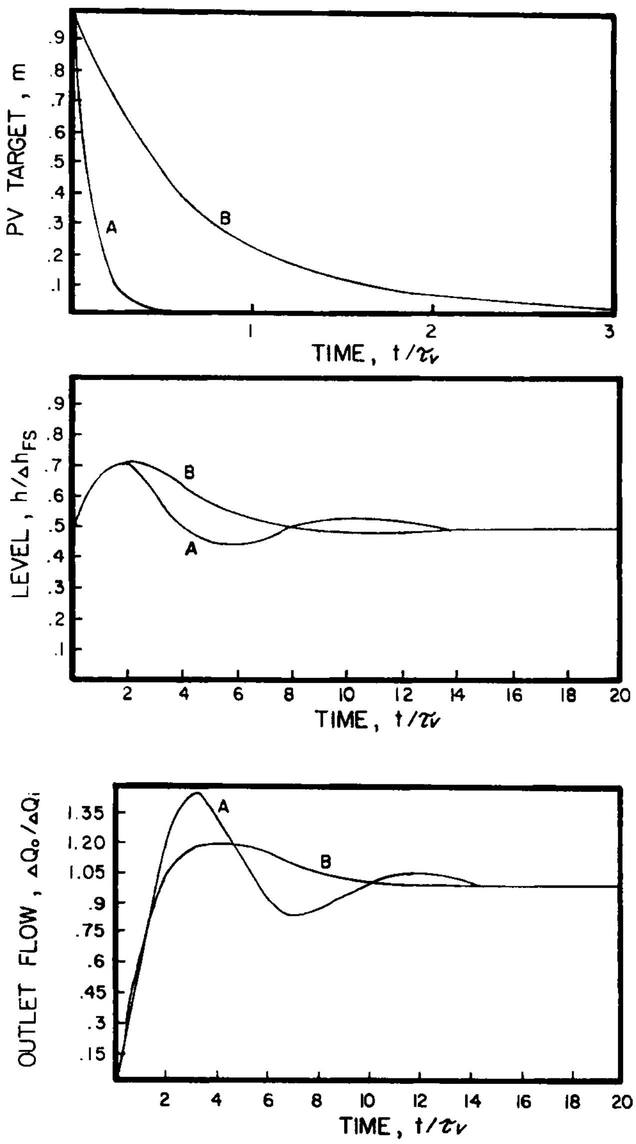

Figure 6. Effect of DMC tuning parameters Sampling period = .05 min. Input horizon = 0.25 min.

provides excellent flow filtering (MRCO = 1.42 m^3/min/min compared to 1.52 m^3/min/min for the OPC using a sampling period of 0.05 min) but the outlet flow overshoot (44%) is unacceptable and the level and outlet flow responses are very oscillatory. This type of response is primarily due to the short output horizon and tight projection vector target area, which requires the DMC algorithm to force the level back to the steady state setpoint in a short time period. When the output horizon is increased from 10 to 30 sampling periods (simulation *B*), keeping the sampling period at 0.05 min, the overshoot is reduced and the oscillations disappear. The MRCO is increased, however, to 1.67 m^3/min/min or 10% above the optimal predictive controller MRCO. When this DMC controller is compared with the optimal predictive controller for slowly varying sinusoidal disturbances, the DMC controller amplifies the disturbance while the OPC provides good flow filtering. Again, the primary reason for the poor DMC performance is that the projection vector target area is too "tight," in that it requires the controller to return the level to the steady state very quickly. A wider target area would allow small deviations from steady state without changing the outlet flow rate. A set of tuning parameters was chosen with a wider projection vector target area, keeping the sampling period, output horizon, and input horizon the same. With the wider target area it was difficult to achieve a maximum peak height of 0.7 for a 100% step in inlet flow even for the lowest values of the input penalty parameter. In order to facilitate the trial and error tuning and allow a comparison with the optimal predictive controller, DMC tuning parameters were chosen which would allow a MPH of 0.8 for a 50% step change in inlet flow. These tuning parameters, which proved to be reasonable for step disturbances in terms of the MRCO and also filtered sinusoidal disturbances, are shown in Figure 7.

Although the level projection vector target area provides the "averaging" characteristics of the DMC controller, it introduces new tuning parameters, thereby complicating the DMC tuning procedure. There may be other ways of modifying the DMC algorithm to achieve averaging control without increasing the difficulty of tuning the controller. In addition, a recent extension of the DMC algorithm, called quadratic dynamic matrix control, or QDMC (Cutler et al., 1983), explicitly considers constraints on the controlled and manipulative variables. The QDMC algorithm would probably be more appropriate for the averaging level control problem since the level constraint could be incorporated directly and would not be violated as it is with DMC for some choices of the tuning parameters. As a result, tuning the QDMC controller would probably be simpler.

```
SAMPLING PERIOD            =  .05 MIN.
OUTPUT VARIABLE HORIZON    =   30 SAMPLING PERIODS
INPUT VARIABLE HORIZON     =    5 SAMPLING PERIODS
PENALTY ON INPUT CHANGES   =  6.5

PROJECTION VECTOR TARGET AREA (FUNNEL):
```

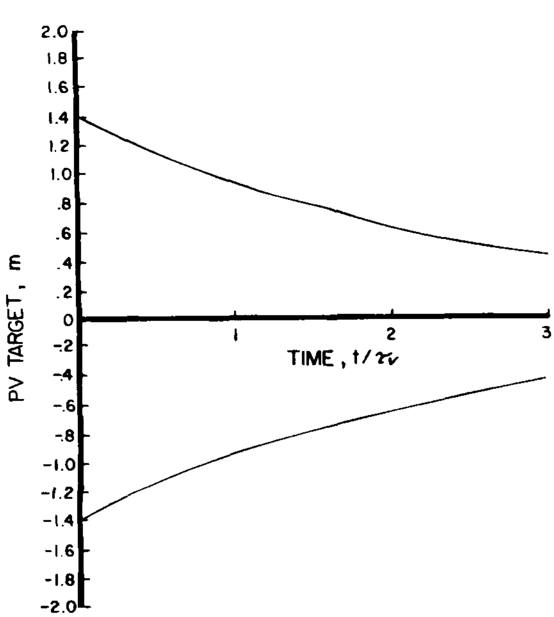

Figure 7. Selected DMC tuning parameters.

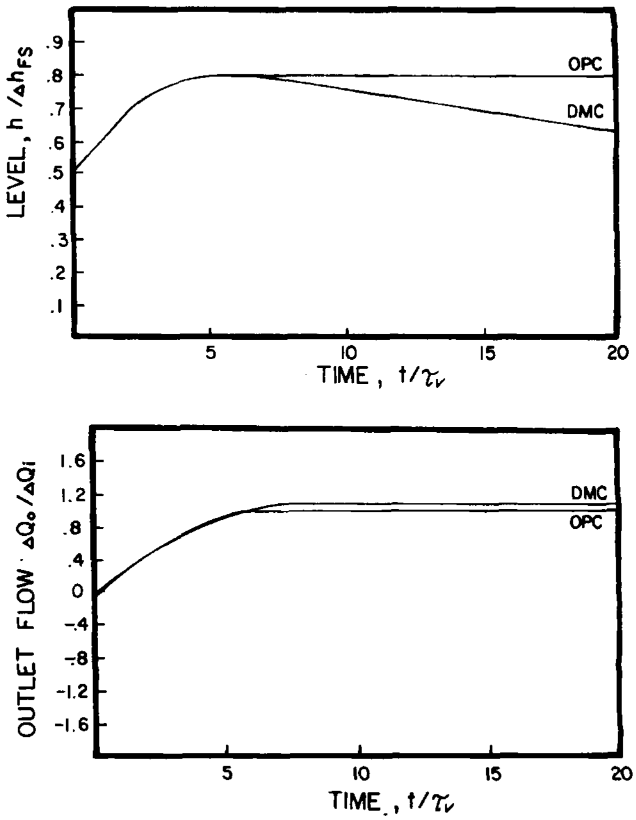

Figure 8. Dynamic matrix control vs. optimal predictive control.
MPH = 80%; 50% step in inlet flow.

Tuning the Optimal Predictive Controller

Simulations using the optimal predictive controller were run using a MPH of 0.8 for a 50% step in inlet flow. A sampling period of 0.05 min was also used for this controller. The gain and reset time determined from Eqs. 25 and 26 were 0.153 and 9.0 min, respectively.

Step Disturbances

Using the test system parameters given in Table 1 and the DMC and OPC settings described above, simulations for 10, 50, and 100% step disturbances were carried out. Measurement noise and bias were ignored in these simulations. Results from these simulations are shown in Figures 8–10. Figure 8 shows the results for the 50% step for which the controllers were designed. As can be seen from this figure, the initial outlet flow responses are very similar for the two controllers; the MRCO for the optimal controller is 0.23 m^3/min/min and 0.25 m^3/min/min for the DMC controller. The DMC controller results in slightly greater overshoot, which tends to bring the level back down more quickly than the OPC does. Because of the effective "dead zone" in the

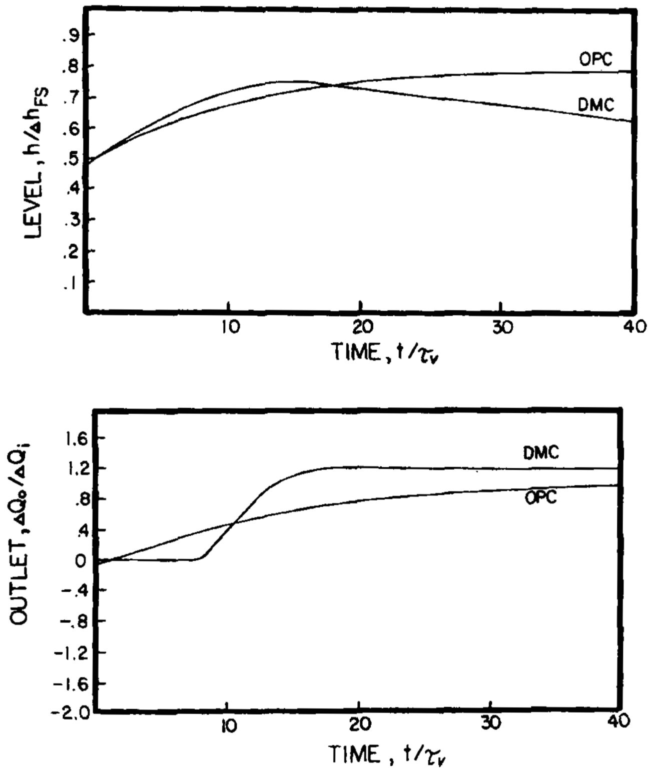

Figure 9. Dynamic matrix control vs. optimal predictive control.
MPH = 80%; 10% step in inlet flow.

projection vector target area (Figure 7) the level overshoots the steady state value before bringing the level back to the steady state. As shown in Figure 9, for the 10% step disturbance the optimal predictive controller provides much better flow filtering than the DMC controller. This is because the loose projection vector target area does not require the DMC to take control action until the level deviates significantly from the steady state value. For this case, the MRCO of the DMC controller is almost twice the MRCO for the OPC even though the amount of tank used in the transient is approximately the same. A 100% step disturbance, shown in Figure 10, causes the DMC controller to violate the MPH specification, while the OPC is able to satisfy the MPH constraint by increasing the MRCO. One of the advantages of the optimal predictive controller is that the level constraint will always be satisfied, which is not necessarily true for the DMC controller.

Sinusoidal Disturbances

Since previous results using the dynamic matrix controller for averaging level control (Cutler et al., 1983) demonstrated excellent filtering performance, particularly for oscillatory disturbances in inlet flow, the DMC and OPC were also compared for sinusoidal variations in the inlet flow. The tuning parameters used in the simulations were identical to those used for the design case, a MPH = 0.8 for a 50% step in inlet flow. A range of frequencies was considered: 1.33×10^{-3} Hz, 2.65×10^{-3} Hz, and 8.84×10^{-3} Hz. At each frequency, the amplitude that would cause the level to reach the MPH specification with no control

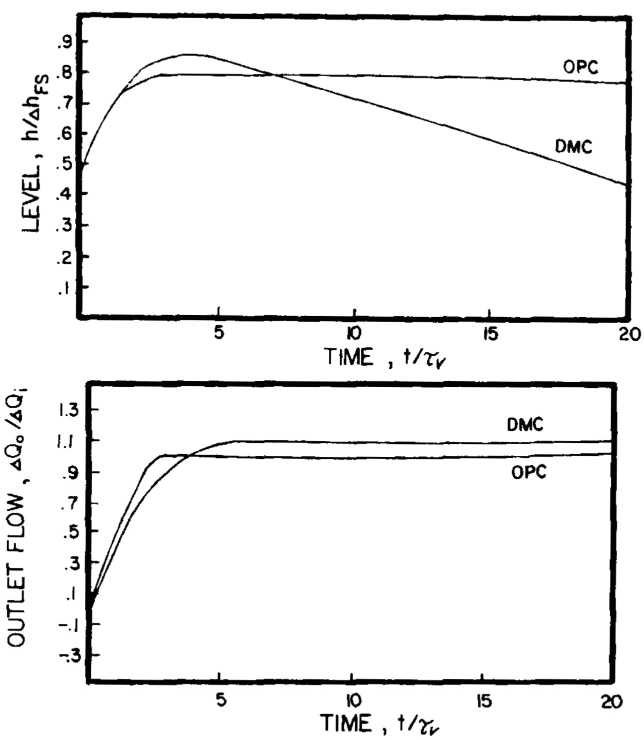

Figure 10. Dynamic matrix control vs. optimal predictive control.
MPH = 80%; 100% step in inlet flow.

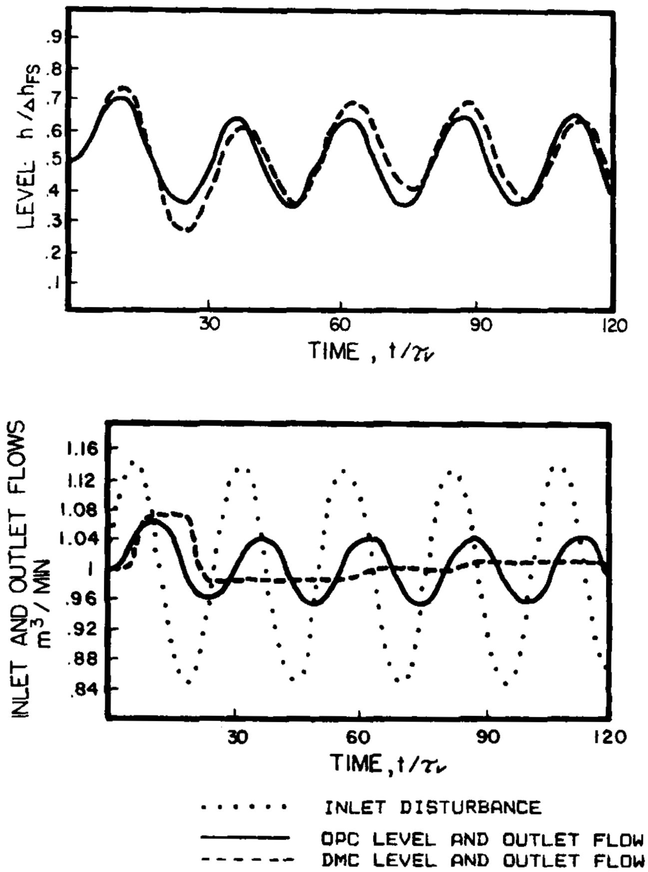

Figure 11. Dynamic matrix control vs. optimal predictive control.
Sinusoidal disturbance in inlet flow:
Frequency = 1.33×10^{-3} Hz.
Amplitude = 0.15 m^3/min.

action was calculated. Sinusoidal disturbances with this amplitude as well as a larger and smaller amplitude were considered.

At the lower frequencies, 1.33×10^{-3} Hz and 2.65×10^{-3} Hz, the filtering performance of the DMC is superior to the OPC outlet flow response. This is true for cases in which the amplitude of the disturbance is such that it would cause the level to reach the MPH specifications if no control action were taken, as well as for smaller amplitude disturbances. Typical DMC and OPC results are shown in Figure 11 for a 1.33×10^{-3} Hz, 0.15 m^3/min amplitude disturbance. The filtering action of the DMC controller is much better even though the level responses of the two controllers are similar and the level constraints are satisfied by both controllers. For a larger disturbance, 0.225 m^3/min, at the same frequency, the DMC violates the level constraints and results in poor flow filtering, while the OPC is able to provide fairly good filtering while keeping the level in bounds. Results for this case are shown in Figure 12. Thus it appears that when the DMC controller is forced with a larger disturbance than it was designed for, a deterioration in filtering performance results and the level constraints may be violated.

For the higher frequency disturbance, 8.84×10^{-3} Hz, the level and outlet flow results are very similar. Both controllers do a fairly good job in filtering the input disturbance. A 1.0 m^3/min amplitude disturbance is reduced by 60–70%, while a 0.5 m^3/min disturbance is reduced by 85–90%. A larger amplitude disturbance at this frequency could not be considered since the inlet flow would have to drop below zero.

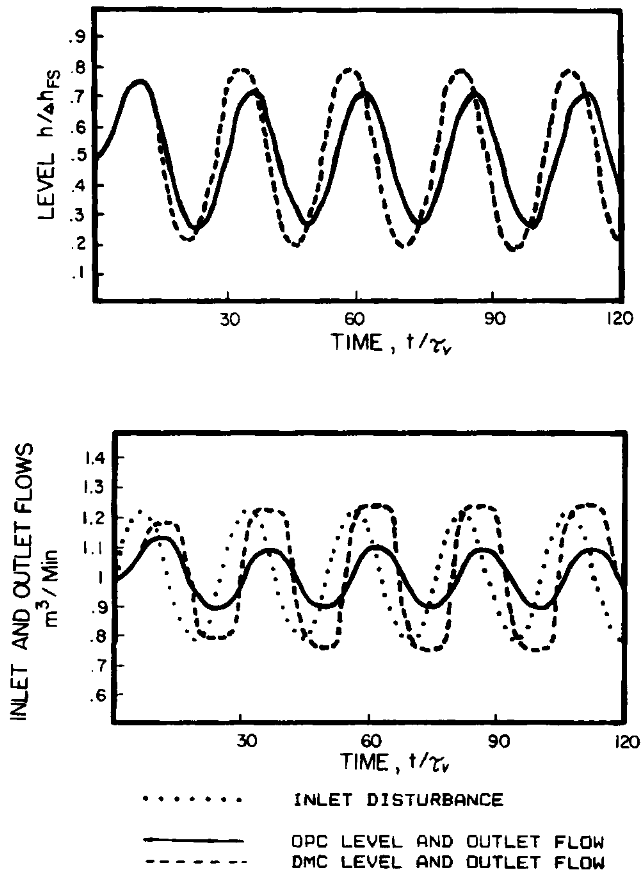

Figure 12. Dynamic matrix control vs. optimal predictive control.
Sinusoidal disturbance in inlet flow:
Frequency = 1.33×10^{-3} Hz.
Amplitude = 0.225 m^3/min.

Dead Time

In these simulations, the dead time associated with the inlet flow disturbance was assumed to be negligible. Although this would be the case for a surge tank averaging level control problem, it may be unrealistic for a reboiler or bottoms level control problem. The DMC algorithm can easily handle the dead time, and an improvement in filtering performance can result. For example, when a positive step in the inlet flow is measured, the DMC controller can begin to increase the outlet flow rate before the disturbance has an effect on the level. This action causes the level to drop during the dead time, thereby increasing the effective tank volume that can be used for filtering the disturbance. Some of the improvements achieved by DMC over classical control, shown in Figure 1, appear to be the result of measuring the disturbance before it affects the level. The OPC approach can be extended to include a dead time in the inlet flow disturbance. The resulting algorithm, however, is much more complex than Eq. 19. Because of this complexity it may be simpler to use a DMC approach when dead time is present. Further work is required to determine the benefits and/or problems resulting from dead time in averaging level control problems as well as the advantages/disadvantages of a DMC vs. an OPC approach to such cases.

ACKNOWLEDGMENT

Financial support from the National Science Foundation under Grant CPE-8025301 is gratefully acknowledged. K. McDonald is grateful to the Graduate School at the University of Maryland for a research fellowship. The authors wish to acknowledge helpful discussions with Henry Lau, David Prett, and Charles Herzog at Shell Development Co.

NOTATION

A = cross-sectional area of tank
B = magnitude of step disturbance in inlet flow
h = level in tank
K_c = controller gain
MPH = maximum peak height
MRCO = maximum rate of change in outlet flowrate
t = time
q = flow rate
α = slope of optimal outlet flow, defined in Eq. 5
τ_I = reset time of controller
τ_V = holdup time of the tank = $\Delta hA/\Delta Q_{max}$

Subscripts

s = steady state
i = inlet
o = outlet

Superscripts

$\hat{}$ = deviation variable
$*$ = optimal

APPENDIX. PROOF OF EQ. 5

Although the quantity to be minimized, MRCO, is usually defined as the time-derivative of the outlet flow, it seems desirable not to restrict ourselves to the case in which this outlet flow is an everywhere differentiable function of time (it will in fact turn out that the most "natural" one in the family of solutions to the optimal control problem presents a "corner"). A natural generalization of the classical definition of MRCO seems to be

$$\sup_{t \neq t'} \left| \frac{q_o(t) - q_o(t')}{t - t'} \right| \tag{A1}$$

which is identical to the classical definition whenever q_o is everywhere differentiable. The natural class of admissible controls to consider is the class of functions such that Eq. A1 is finite, i.e., the class of Lipschitz continuous functions, or more particularly for our problem, functions that are Lipschitz continuous on $R^+ = [0, \quad]$ (note that, for all such functions, the solution to Eq. 4 is well defined).

The control problem can be stated as follows.

(P): Given scalars h_{min}, h_s, B, q_s satisfying

$$h_{min} \leq h_s \leq h_{max}, B > 0, q_s > 0 \tag{A2}$$

find a control q_o, Lipschitz continuous on R^+, so as to minimize over the class of such functions the objective

$$\sup_{\substack{t,t' \epsilon R^+ \\ t \neq t'}} \left| \frac{\dot{q}_o(t) - \dot{q}_o(t')}{t - t'} \right| \tag{A3}$$

subject to the conditions

$$\dot{q}_o(0) = q_s \tag{A4}$$

$$h_{min} \leq h_s + \frac{1}{A}\int_o^t (B + q_s - \dot{q}_o(\tau))d\tau \leq h_{max} \forall t \geq 0 \tag{A5}$$

$$\dot{q}_o(t) \geq 0 \quad \forall t \geq 0 \tag{A6}$$

The following lemma solves a "stripped down" version of the above problem.

Lemma.

Consider the problem (P')

$$\underset{u \epsilon \mathcal{u}}{\text{minimize}} \sup_{\substack{t,t' > 0 \\ t \neq t'}} \left| \frac{u(t) - u(t')}{t - t'} \right| \tag{A7}$$

where

$$\mathcal{u} \underline{\underline{\Delta}} [u{:}R^+ \quad R | u \text{ is Lipschitz continuous over } R^+,$$

$$u(o) = U, 0 \leq \int_o^t u(\tau)d\tau \leq M \forall t \geq 0] \tag{A8}$$

where $U > 0$ and $M > 0$ are given.

This problem admits a solution u^* given by

$$u^*(t) = \begin{cases} U - \dfrac{U^2}{2M}t & t \epsilon [0, \dfrac{2M}{U}] \\ 0 & t > \dfrac{M}{U} \end{cases} \tag{A9}$$

Moreover a function $\hat{u}$, Lipschitz continuous over R^+, is a solution to (P') if and only if the following three conditions hold

$$\hat{u}(t) = u^*(t) \quad \forall t \;\epsilon [0, \frac{2M}{U}] \tag{A10}$$

$$-M \leq \int_{\frac{2M}{U}}^t \hat{u}(\tau)d\tau \leq 0 \quad \forall t \geq \frac{2M}{U} \tag{A11}$$

$$\left| \frac{\hat{u}(t) - \hat{u}(t')}{t - t'} \right| \leq \frac{U^2}{2M} \forall t,t' \geq \frac{2M}{U}, t \neq t' \tag{A12}$$

Proof

First, u^* is clearly admissible. In particular, $\int_o^t u^*(\tau)\,d\tau$ is nondecreasing when t increases, vanishes at $t = 0$, and

$$\int_o^t u^*(\tau)d\tau = M \quad t \geq \frac{2M}{U} \tag{A13}$$

$$J(u^*) = \sup_{\substack{t,t' \geq 0 \\ t \neq t'}} \left| \frac{u^*(t) - u^*(t')}{t - t'} \right| = \frac{U^2}{2M} \tag{A14}$$

We now show, by contradiction, that u^* is a solution to (P')

Suppose $\tilde{u} \;\epsilon\; \mathcal{u}$ is such that

$$\sup_{\substack{t,t' \epsilon R^+ \\ t \neq t'}} \left| \frac{\tilde{u}(t) - \tilde{u}(t')}{t - t'} \right| < \sup_{\substack{t,t' \epsilon R^+ \\ t \neq t'}} \left| \frac{u^*(t) - u^*(t')}{t - t'} \right| = \frac{U^2}{2M} \tag{A15}$$

i.e., $\exists \beta < \dfrac{U^2}{2M}$

$$\left| \frac{\tilde{u}(t) - \tilde{u}(t')}{t - t'} \right| \leq \beta \forall t,t' \geq 0, \quad t \neq t' \tag{A16}$$

In particular, if we pick $t' = 0$ (and $t > 0$)

$$\left| \frac{\tilde{u}(t) - U}{t} \right| \leq \beta \;\forall t \geq 0 \tag{A17}$$

and, in particular,

$$\tilde{u}(t) \geq U - \beta t > U - \frac{U^2}{2M}t \;\forall t > 0 \tag{A18}$$

Hence

$$\int_o^{\frac{2M}{U}} \tilde{u}(\tau)d\tau > \int_o^{\frac{2M}{U}} (U - \frac{U^2}{2M})d\tau$$

$$= 2M - \frac{U^2}{2M} \frac{\left(\frac{2M}{2}\right)^2}{2} = M \tag{A19}$$

which shows that $\tilde{u}$ is not admissible. This contradiction proves that u^* is indeed a solution. Now, let $\hat{u}$ be any solution to (P'). Since u^* is a solution as well, it comes from Eq. A14 that

$$J(\hat{u}) = J(u^*) = \frac{U^2}{2M} \tag{A20}$$

and hence Eq. A12 must hold. By an argument analogous to the one that led to Eq. A18 we obtain

$$\hat{u}(t) \geq U - \frac{U^2}{2M}t \quad \forall t \geq 0 \tag{A21}$$

But this implies that

$$\hat{u}(t) = U - \frac{U^2}{2M}t \quad \forall t \in [0, \frac{2M}{U}] \tag{A22}$$

since, if there were a $\underline{t} \in [0, \frac{U^2}{2M}$ such that

$$\hat{u}(\underline{t}) > U - \frac{U^2}{2M}\underline{t} \tag{A23}$$

it would imply, in view of the continuity of the functions involved, that

$$\int_0^{\frac{2M}{U}} \hat{u}(\tau)d\tau > \int_0^{\frac{2M}{U}} (U - \frac{U^2}{2M}\tau)d\tau = M \tag{A24}$$

which would contradict the fact that $\hat{u}$ is admissible. Hence Eq. A22 holds and Eq. A10 is proven. Finally Eq. A11 must hold in order for $\hat{u}$ to be admissible since, for $t > 2M/U$, using Eqs. A10 and A13

$$\int_0^t \hat{u}(\tau)d\tau = \int_0^{\frac{2M}{U}} \hat{u}(\tau)d\tau + \int_{\frac{2M}{U}}^t \hat{u}(\tau)d\tau \tag{A25}$$

$$= M + \int_{\frac{2M}{U}}^t \hat{u}(\tau)d\tau \tag{A26}$$

and since admissibility requires

$$\int_0^t \hat{u}(\tau)d\tau \quad \in [0,M] \quad \forall t > 0 \tag{A27}$$

Hence any solution $\hat{u}$ to (P') must satisfy Eqs. A10, A11, and A12. Conversely, if $\hat{u}$ is such that Eqs. A10, A11, and A12 hold, it is easily checked that $\hat{u}$ is admissible, since, from Eqs. A10, A9, and A12 one gets

$$\sup_{\substack{t,t' \in R^+ \\ t \neq t'}} \left| \frac{\hat{u}(t) - \hat{u}(t')}{t - t'} \right| = \max \left\{ \sup_{\substack{t,t' \in [0, \frac{2M}{U}] \\ t \neq t'}} \left| \frac{u^*(t) - u^*(t')}{t - t'} \right|, \sup_{\substack{t,t' > \frac{2M}{U} \\ t \neq t'}} \left| \frac{\hat{u}(t) - \hat{u}(t')}{t - t'} \right| \right\} \tag{A28}$$

$$= \frac{U^2}{2M} \tag{A29}$$

The following theorem about problem (P) is a direct consequence of the above lemma.

Theorem

There is a solution to problem (P) given by

$$\dot{q}_o(t) = \begin{cases} q_s + \alpha t & \text{for } t \in [0, t_{max}] \\ q_s + \alpha t_{max} = q_s + B_{max} & \text{for } t > t_{max} \end{cases} \tag{A30}$$

with

$$\alpha = \frac{B^2}{2A(h_{max} - h_s)}, \quad t_{max} = \frac{2A(h_{max} - h_s)}{B}$$

Moreover, a flow outlet profile $q_o(t)$ is a solution to (P) if and only if

$$q_o(t) = \dot{q}_o(t) \quad \forall t \in [0, t_{max}] \tag{A31}$$

and, $\forall t, t' \geq t_{max}, t \neq t'$

$$-(h_{max} - h_{min}) \leq \frac{1}{A} \int_{t_{max}}^t (B + q_s - \dot{q}_o(\tau))d\tau \leq 0 \tag{A32}$$

$$\left| \frac{q_o(t) - q_o(t')}{t - t'} \right| \leq \alpha \tag{A33}$$

$$q_o(t) \geq 0 \tag{A34}$$

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Manuscript received Mar. 27, 1984, and revision received Feb. 5, 1985.